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Note

Boundary layer due to the axisymmetric spreading of material on the surface of a fluid

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ABSTRACT

The effect of the axisymmetric spreading of a layer of material (oil or solid particles) on the surface of a viscous fluid is studied. Assuming high Reynolds numbers, the boundary layer equation is derived and solved for general power law surface velocities. The composite streamlines show sharp turns near the surface.

1. Introduction

Consider a layer of lighter, higher viscosity fluid (or solid particles) spreading from a source in a layer laterally on top of the surface of a heavier, lower viscosity fluid (Fig. 1). The lower viscosity bottom fluid is being dragged by the top fluid through surface stress, but has little effect on the top fluid. The problem models spreading of oil on water. In the case the top fluid spreads with constant thickness, the velocity imparted to the lower fluid is proportional to the inverse of the distance to the source. Such a problem was solved by Wang [1], who obtained a rare exact solution to the Navier–Stokes equations. However, in some cases the thickness of the top fluid layer may not be constant, but varies with the distance from the source. This affects the boundary velocity imparted to the lower fluid. The purpose of the present paper is to study this generalized problem. Although exact solutions do not exist, we seek similarity solutions for the boundary layer equations at high Reynolds numbers.

Using cylindrical coordinates (r', z') as shown in Fig. 1, the boundary layer is near the interface at $z' = 0$. At a typical radial length scale L let the thickness be H_0. The local velocity is then

$$V_0 = \frac{Q}{2\pi L H_0} \tag{1}$$

where Q is the flow rate at the source. Normalize all lengths by L, all velocities by V_0. Eq. (1) in normalized form gives

$$V(r) = \frac{1}{rH(r)} \tag{2}$$

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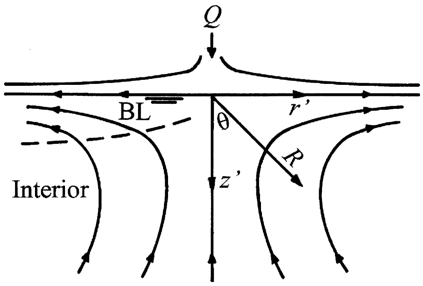

Fig. 1. Axisymmetric spreading of material on the surface of a fluid. BL indicates boundary layer.

We consider a power law thickness $H(r) = r^{-m-1}$, m being any number. Notice that the thickness of the top layer decreases with radius if $m > -1$, and increases with radius if $m < -1$. If the top layer is thin, the interface velocity is given by the mean velocity

$$V = r^m \tag{3}$$

Next consider the lower fluid. Let the Reynolds numbers be high, such that

$$\frac{V_0 L}{\nu} \equiv \frac{1}{\varepsilon^2} \gg 1 \tag{4}$$

Here ν is the kinematic viscosity of the (lower) fluid.

The problem is analogous to the flow due to an axisymmetric stretching boundary. The two-dimensional stretching boundary layer has been studied by many authors, notably Kuiken [2], Banks [3], Ingham and Pop [4], Weidman and Magyari [5]. The axisymmetric stretching boundary layer, only for $m = 1$, was solved by Wang [6], with some analyses given by Wang [7].

2. Formulation

The normalized continuity and Navier–Stokes equations in axisymmetric cylindrical coordinates are

$$(ru)_r + (rw)_z = 0 \tag{5}$$

$$uu_r + wu_z = -p_r + \varepsilon^2\left(u_{rr} + u_r/r + u_{zz} - u/r^2\right) \tag{6}$$

$$uw_r + ww_z = -p_z + \varepsilon^2(w_{rr} + w_r/r + w_{zz}) \tag{7}$$

Here (u, w) are velocities in (r, z) directions respectively, and p is the pressure normalized by ρV_0^2, ρ being the density. In the boundary layer, let

$$z = \varepsilon\xi, \qquad w = \varepsilon\upsilon \tag{8}$$

The zeroth order boundary layer approximation of Eqs. (5)–(7) gives

$$(ru)_r + (r\upsilon)_\xi = 0 \tag{9}$$

$$uu_r + \upsilon u_\xi = -p_r + u_{\xi\xi} \tag{10}$$

$$p_\xi = 0 \tag{11}$$

Since there is no zeroth order interior flow outside the boundary layer, we can set the pressure to be a constant. Guided by Eqs. (3), (9), (10) we use the similarity transform

$$u = r^m f'(\eta), \quad \eta = r^{(m-1)/2}\xi \tag{12}$$

$$\upsilon = -r^{(m-1)/2}\left[\frac{(m+3)}{2}f + \frac{(m-1)}{2}\eta f'\right] \tag{13}$$

The boundary layer equations reduce to

$$f''' + \frac{(m+3)}{2}ff'' - m\left(f'\right)^2 = 0 \tag{14}$$

The boundary conditions are

$$f(0) = 0, \qquad f'(0) = 1 \tag{15}$$

$$f'(\infty) = 0 \tag{16}$$

Eq. (14) is quite different from the two-dimensional boundary layer equations studied by the previous authors. From Eqs. (9), (12) a stream function is obtained

$$\psi = \varepsilon r^{(m+3)/2} f(\eta) \tag{17}$$

3. Some analyses

Lemma 1. *The function f decays exponentially for large η.*

Proof. Eq. (16) suggests $f \sim C + \varphi(\eta)$ for large η, where $\varphi(\eta)$ decay to zero and

$$f(\infty) = C \tag{18}$$

C being a constant. Let

$$a = \frac{m+3}{2} \tag{19}$$

Eq. (14) becomes

$$f''' + aff'' - m(f')^2 = 0 \tag{20}$$

For large η Eq. (19) linearizes to

$$\varphi''' + aC\varphi'' = 0 \tag{21}$$

In order for φ to decay, $aC > 0$, and φ decays as $e^{-aC\eta}$. $\square$

Lemma 2. $f \geqslant 0$.

Proof. If $C < 0$ then from Lemma 1 $a < 0$ and Eq. (19) gives $m < -3$. From Eq. (15) f rises to a maximum since Eq. (18) shows f must dip below zero. Thus there is an inflection point where $f' < 0$, $f'' = 0$, $f''' > 0$. Eq. (20) then shows $m > 0$ which is a contradiction.

If $C > 0$ from Lemma 1 $a > 0$ and $m > -3$. If f dips below zero, at the first inflection point similar arguments show $m > 0$. However f must rise up eventually, since $C > 0$. At the last inflection point $f' > 0$, $f'' = 0$, $f''' < 0$. From Eq. (20) $m < 0$ which is a contradiction. We conclude f is never negative. $\square$

Lemma 3. $m > -3/2$.

Proof. Multiply Eq. (20) by f and integrate from zero to infinity. Using integration by parts we find

$$\frac{1}{2} - (2a+m)\int_0^\infty f(f')^2\,d\eta = 0 \tag{22}$$

Since by Lemma 2 f is non-negative, Eq. (22) shows $2a + m > 0$ or $m > -3/2$. $\square$

4. The boundary layer solution

Wang [1] found the exact solution to the axisymmetric boundary layer Eq. (14) for $m = -1$, or the constant thickness top layer, viz.

$$f = \sqrt{2}\tanh(\eta/\sqrt{2}) \tag{23}$$

giving $f''(0) = 0$ and $f(\infty) = C = \sqrt{2}$. Using the Runge–Kutta algorithm and a shooting scheme, numerical solutions for other $m > -3/2$ values can be found. Table 1 shows the initial values and the final values.

The results for the $m = 1$ case agree with the axisymmetric stretching solution of Wang [6]. Fig. 2 shows the boundary layer solutions for various m values. The boundary layer thickness is about $\eta \sim 5$.

Table 1
Some initial and final values.

m	-1	-0.5	0	0.5	1	1.5	2	3
$f''(0)$	0	−0.47861	−0.76859	−0.98962	−1.1737	−1.3343	−1.4783	−1.7321
C	$\sqrt{2}$	1.09684	0.93307	0.82735	0.75150	0.69353	0.64731	0.57734

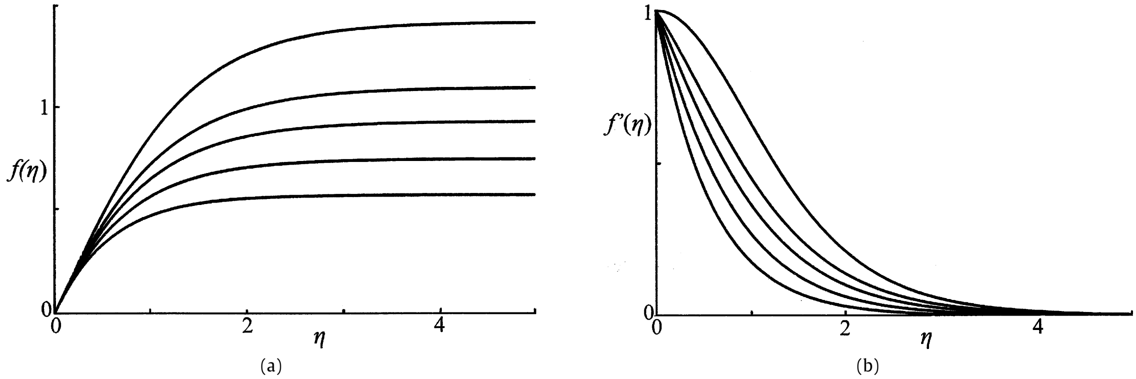

Fig. 2. Boundary layer profiles: (a) $f(\eta)$, (b) $f'(\eta)$. From top: $m = -1, -0.5, 0, 1, 3$.

5. The interior flow

The flow outside the boundary layer is of importance since it governs the convective transport of the lower fluid. From Eq. (17) we find at large distances

$$\psi = \varepsilon r^{(m+3)/2} C \tag{24}$$

which drives the interior flow. Eqs. (5)–(7) show the interior flow is potential for the first two orders. In axisymmetric spherical polar coordinates the velocity potential Φ is governed by

$$\nabla^2 \Phi = \left(R^2 \Phi_R\right)_R + \frac{1}{\sin\theta}(\sin\theta \Phi_\theta)_\theta = 0 \tag{25}$$

Here $R = \sqrt{r^2 + z^2}$ is the polar radial coordinate and θ is the angle from the z axis (Fig. 1).

Let $\varsigma = \cos\theta$ and

$$\Phi = \lambda R^\beta \chi(\varsigma) \tag{26}$$

where λ and β are constants to be determined. Eq. (25) reduces to the Legendre equation

$$\left(1 - \varsigma^2\right)\chi'' - 2\varsigma\chi' + \beta(\beta + 1)\chi = 0 \tag{27}$$

The relation between the velocity potential and the stream function is [8]

$$\Phi_R = \frac{1}{R^2 \sin\theta}\Psi_\theta, \qquad \Phi_\theta = \frac{-1}{R\sin\theta}\Psi_R \tag{28}$$

The stream function for the interior is integrated

$$\Psi = \lambda\beta R^{\beta+1} \int_1^\varsigma \chi(\varsigma)\, d\varsigma, \quad \beta \neq 0 \tag{29}$$

Matching the boundary layer stream function Eq. (24) and interior stream function Eq. (29) and using the integral of Eq. (27)

$$\left(1 - \varsigma^2\right)\chi' + \beta(\beta + 1)\int_1^\varsigma \chi\, d\varsigma = 0 \tag{30}$$

we find

$$\beta = \frac{m+1}{2}, \qquad \lambda = -\varepsilon C\left(\frac{m+3}{2}\right) \tag{31}$$

$$\chi'(0) = 1 \tag{32}$$

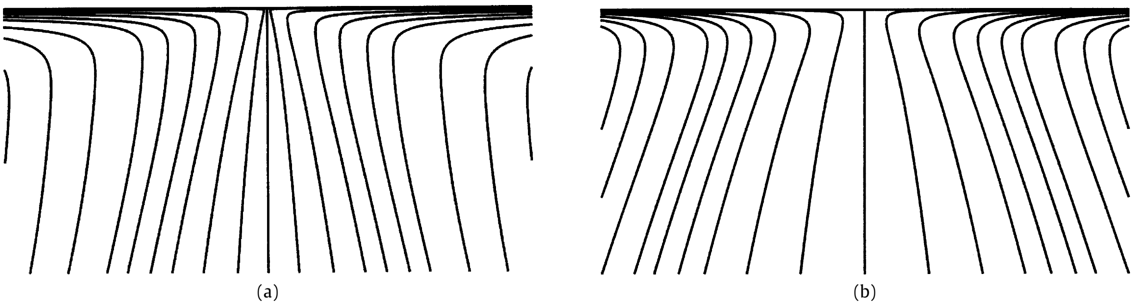

Fig. 3. Composite streamlines. The domain is $0 \leqslant (r, z) \leqslant 2$; (a) $m = -0.5$, (b) $m = 1$. From center: $\psi = 0$, 0.001, 0.01, 0.03, 0.05, 0.075, 0.1, 0.15, 0.2, 0.25, 0.3.

The bounded solution to Eq. (27) and satisfying Eq. (32) is the odd Legendre function [9]

$$\chi = \varsigma - \frac{(\beta-1)(\beta+2)}{3!}\varsigma^3 + \frac{(\beta-1)(\beta-3)(\beta+2)(\beta+4)}{5!}\varsigma^5 - \cdots \tag{33}$$

The series terminates if β is a positive odd integer. For $\beta = 0$ or $m = -1$, the solution is different, being [1]

$$\Psi = \sqrt{2}R(1-\varsigma), \qquad \Phi = \sqrt{2}\ln\bigl[R(1+\varsigma)\bigr] \tag{34}$$

A composite solution can be constructed [10]

$$\begin{aligned} \psi_c &= \psi + \Psi - \text{common part} \\ &= \varepsilon r^{(m+3)/2}\bigl[f(\eta) - C\bigr] + \varepsilon C R^{(m+3)/2}\bigl(1-\varsigma^2\bigr)\chi'(\varsigma) + O\bigl(\varepsilon^2\bigr) \end{aligned} \tag{35}$$

Some composite streamlines are shown in Fig. 3. Notice that as m increases, the distances between streamlines become wider, showing a relatively lower velocity. Note also the small curvatures of the streamlines near the edge of the boundary layer.

6. Discussion

The spreading of a layer of material on the surface of a low viscosity fluid is now delineated. We assume the spreading material imparts a surface velocity proportional to r^m to the lower fluid. The value of m depends on how the top material is delivered. If it is from a point source with constant rate then m depends on the thickness variations of the spreading layer as discussed in Section 1. There are other causes for a variable surface velocity. For example a uniform shower of material collecting in a uniform layer would give $m = 1$.

Our basic equation, Eq. (14), is an axisymmetric similarity boundary layer equation. Since the flow is induced by the tangential surface velocity, the equation differs from pressure driven similarity flows [11,12]. On the other hand, it also differs from two-dimensional stretching problems. There are, however, some analogies. When $m = -1$ the tanh solution Eq. (23) was found by Schlichting [11] for a two-dimensional jet. When $m = 0$ the solution is analogous to the two-dimensional extrusion of a plate studied by Sakiadis [13]. Section 3 shows no similarity solutions exist for $m \leqslant -3/2$.

The composite streamlines are of interest. Unlike stagnation flow, the fluid makes a sharp turn when it is entrained by the boundary layer. This is especially true for negative m values. Physically this means any denser, convected particles (or organisms) would be centrifuged at the turn and possibly deposited onto the surface material.

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